1911 Encyclopædia Britannica/Calculating Machines

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CALCULATING MACHINES. Instruments for the mechanical performance of numerical calculations, have in modern times come into ever-increasing use, not merely for dealing with large masses of figures in banks, insurance offices, &c., but also, as cash registers, for use on the counters of retail shops. They may be classified as follows:—(i.) Addition machines; the first invented by Blaise Pascal (1642). (ii.) Addition machines modified to facilitate multiplication; the first by G.W. Leibnitz (1671). (iii.) True multiplication machines; Léon Bollés (1888), Steiger (1894). (iv.) Difference machines; Johann Helfrich von Müller (1786), Charles Babbage (1822). (v.) Analytical machines; Babbage (1834). The number of distinct machines of the first three kinds is remarkable and is being constantly added to, old machines being improved and new ones invented; Professor R. Mehmke has counted over eighty distinct machines of this type. The fullest published account of the subject is given by Mehmke in the *Encyclopädie der mathematischen Wissenschaften*, article "Numerisches Rechnen," vol. i., Heft 6 (1901). It contains historical notes and full references. Walther von Dyck's *Catalogue* also contains descriptions of various machines. We shall confine ourselves to explaining the principles of some leading types, without giving an exact description of any particular one.

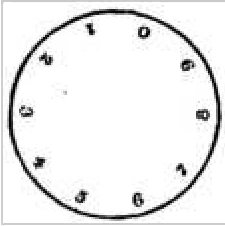

FIG. 1.

Addition machines.

Practically all calculating machines contain a "counting work," a series of "figure disks" consisting in the original form of horizontal circular disks (fig. 1), on which the figures 0, 1, 2, to 9 are marked. Each disk can turn about its vertical axis, and is covered by a fixed plate with a hole or "window" in it through which one figure can be seen. On turning the disk through one-tenth of a revolution this figure will be changed into the next higher or lower. Such turning may be called a "step," *positive* if the next higher and *negative* if the next lower figure appears. Each positive step therefore adds one unit to the figure under the window, while two steps add two, and so on. If a series, say six, of such figure disks be placed side by side, their windows lying in a row, then any number of six places can be made to appear, for instance 000373. In order to add 6425 to this number, the disks, counting from right to left, have to be turned 5, 2, 4 and 6 steps respectively. If this is done the sum 006798 will appear. In case the sum of the two figures at any disk is greater than 9, if for instance the last figure to be added is 8 instead of 5, the sum for this disk is 11 and the 1 only will appear. Hence an arrangement for "carrying" has to be introduced. This may be done as follows. The axis of a figure disk contains a wheel with ten teeth. Each figure disk has, besides, one long tooth which when its 0 passes the window turns the next wheel to the left, one tooth forward, and hence the figure disk one step. The actual mechanism is not quite so simple, because the long teeth as described would gear also into the wheel to the right, and besides would interfere with each other. They must therefore be replaced by a somewhat more complicated arrangement, which has been done in various ways not necessary to describe more fully. On the way in which this is done, however, depends to a great extent the durability and trustworthiness of any arithmometer; in fact, it is often its weakest point. If to the series of figure disks arrangements are added for turning each disk through a required number of steps, we have an addition machine, essentially of Pascal's type. In it each disk had to be turned by hand. This operation has been simplified in various ways by mechanical means. For pure addition machines key-boards have been added, say for each disk nine keys marked 1 to 9. On pressing the key marked 6 the disk turns six steps and so on. These have been introduced by Stettner (1882), Max Mayer (1887), and in the comptometer by Dorr Z. Felt of Chicago. In the comptograph by Felt and also in "Burrough's Registering Accountant" the result is printed.

Modified addition machines.

These machines can be used for multiplication, as repeated addition, but the process is laborious, depending for rapid execution essentially on the skill of the operator.[1] To adapt an addition machine, as described, to rapid multiplication the turnings of the separate figure disks are replaced by one motion, commonly the turning of a handle. As, however, the different disks have to be turned through different steps, a contrivance has to be inserted which can be "set" in such a way that by one turn of the handle each disk is moved through a number of steps equal to the number of units which is to be added on that disk. This may be done by making each of the figure disks receive on its axis a ten-toothed wheel, called hereafter the A-wheel, which is acted on either directly or indirectly by another wheel (called the B-wheel) in which the number of teeth can be varied from 0 to 9. This variation of the teeth has been effected in different ways. Theoretically the simplest seems to be to have on the B-wheel nine teeth which can be drawn back into the body of the wheel, so that at will any number from 0 to 9 can be made to project. This idea, previously mentioned by Leibnitz, has been realized by Bohdner in the "Brunsviga." Another way, also due to Leibnitz, consists in inserting between the axis of the handle bar and the A-wheel a "stepped" cylinder. This may be considered as being made up of ten wheels large enough to contain about twenty teeth each; but most of these teeth are cut away so that these wheels retain in succession 9, 8, ... 1, 0 teeth. If these are made as one piece they form a cylinder with teeth of lengths from 9, 8 ... times the length of a tooth on a single wheel.

In the diagrammatic vertical section of such a machine (fig. 2) FF is a figure disk with a conical wheel A on its axis. In the covering plate HK is the window W. A stepped cylinder is shown at B. The axis Z, which runs along the whole machine, is turned by a handle, and itself turns the cylinder B by aid of conical wheels. Above this cylinder lies an axis EE with square section along which a wheel D can be moved. The same axis carries at E′ a pair of conical wheels C and C′, which can also slide on the axis so that either can be made to drive the A-wheel.

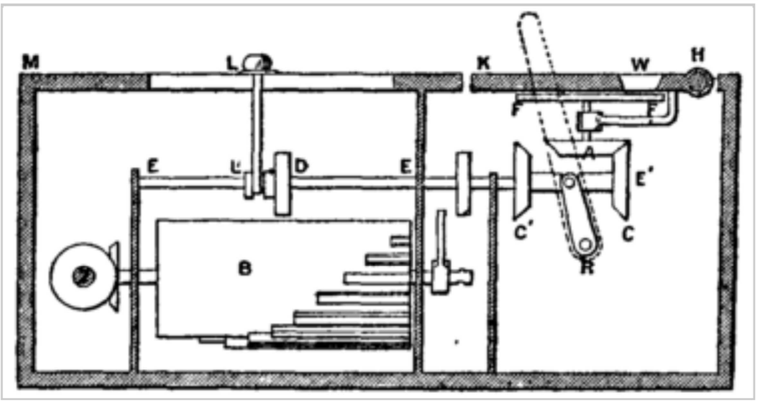

FIG. 2.

The covering plate MK has a slot above the axis EE allowing a rod LL′ to be moved by aid of a button L, carrying the wheel D with it. Along the slot is a scale of numbers 0 1 2 ... 9 corresponding with the number of teeth on the cylinder B, with which the wheel D will gear in any given position. A series of such slots is shown in the top middle part of Steiger's machine (fig. 3). Let now the handle driving the axis Z be turned once round, the button being set to 4. Then four teeth of the B-wheel will turn D and with it the A-wheel, and consequently the figure disk will be moved four steps. These steps will be positive or forward if the wheel C gears in A, and consequently four will be added to the figure showing at the window W. But if the wheels CC′ are moved to the right, C′ will gear with A moving backwards, with the result that four is subtracted at the window. This motion of all the wheels C is done simultaneously by the push of a lever which appears at the top plate of the machine, its two positions being marked "addition" and "subtraction." The B-wheels are in fixed positions below the plate MK. Level with this, but separate, is the plate KH with the window. On it the figure disks are mounted.

This plate is hinged at the back at H and can be lifted up, thereby throwing the A-wheels out of gear. When thus raised the figure disks can be set to any figures; at the same time it can slide to and fro so that an A-wheel can be put in gear with any C-wheel forming with it one "element." The number of these varies with the size of the machine. Suppose there are six B-wheels and twelve figure disks. Let these be all set to zero with the exception of the last four to the right, these showing 1 4 3 2, and let these be placed opposite the last B-wheels to the right. If now the buttons belonging to the latter be set to 3 2 5 6, then on turning the B-wheels all once round the latter figures will be added to the former, thus showing 4 6 8 8 at the windows. By aid of the axis Z, this turning of the B-wheels is performed simultaneously by the movement of one handle. We have thus an addition machine. If it be required to multiply a number, say 725, by any number up to six figures, say 357, the buttons are set to the figures 725, the windows all showing zero. The handle is then turned, 725 appears at the windows, and successive turns add this number to the first. Hence seven turns show the product seven times 725. Now the plate with the A -wheels is lifted and moved one step to the right, then lowered and the handle turned five times, thus adding fifty times 725 to the product obtained. Finally, by moving the piate again, and turning the handle three times, the required product is obtained. If the machine has six B-wheels and twelve disks the product of two six-figure numbers can be obtained. Division is performed by repeated subtraction. The lever regulating the C-wheel is set to subtraction, producing negative steps at the disks. The dividend is set up at the windows and the divisor at the buttons. Each turn of the handle subtracts the divisor once. To count the number of turns of the handle a second set of windows is arranged with number disks below. These have no carrying arrangement, but one is turned one step for each turn of the handle. The machine described is essentially that of Thomas of Colmar, which was the first that came into practical use. Of earlier machines those of Leibnitz, Müller (1782), and Hahn (1809) deserve to be mentioned (see Dyck, *Catalogue*). Thomas's machine has had many imitations, both in England and on the Continent, with more or less important alterations. Joseph Edmondson of Halifax has given it a circular form, which has many advantages.

The accuracy and durability of any machine depend to a great extent on the manner in which the carrying mechanism is constructed. Besides, no wheel must be capable of moving in any other way than that required; hence every part must be locked and be released only when required to move. Further, any disk must carry to the next only after the carrying to itself has been completed. If all were to carry at the same time a considerable force would be required to turn the handle, and serious strains would be introduced. It is for this reason that the B-wheels or cylinders have the greater part of the circumference free from teeth. Again, the carrying acts generally as in the machine described, in one sense only, and this involves that the handle be turned always in the same direction. Subtraction therefore cannot be done by turning it in the opposite way, hence the two wheels C and C′ are introduced. These are moved all at once by one lever acting on a bar shown at R in section (fig. 2).

In the Brunsviga, the figure disks are all mounted on a common horizontal axis, the figures being placed on the rim. On the side of each disk and rigidly connected with it lies its A-wheel with which it can turn independent of the others. The B-wheels, all fixed on another horizontal axis, gear directly on the A-wheels. By an ingenious contrivance the teeth are made to appear from out of the rim to any desired number. The carrying mechanism, too, is different, and so arranged that the handle can be turned either way, no special setting being required for subtraction or division. It is extremely handy, taking up much less room than the others. Professor Eduard Selling of Würzburg has invented an altogether different machine, which has been made by Max Ott, of Munich. The B-wheels are replaced by lazy-tongs. To the joints of these the ends of racks are pinned; and as they are stretched out the racks are moved forward 0 to 9 steps, according to the joints they are pinned to. The racks gear directly in the A-wheels, and the figures are placed on cylinders as in the Brunsviga. The carrying is done continuously by a train of epicycloidal wheels. The working is thus rendered very smooth, without the jerks which the ordinary carrying tooth produces; but the arrangement has the disadvantage that the resulting figures do not appear in a straight line, a figure followed by a 5, for instance, being already carried half a step forward. This is not a serious matter in the hands of a mathematician or an operator using the machine constantly, but it is serious for casual work. Anyhow, it has prevented the machine from being a commercial success, and it is not any longer made. For ease and rapidity of working it surpasses all others. Since the lazy-tongs allow of an extension equivalent to five turnings of the handle, if the multiplier is 5 or under, one push forward will do the same as five (or less) turns of the handle, and more than two pushes are never required.

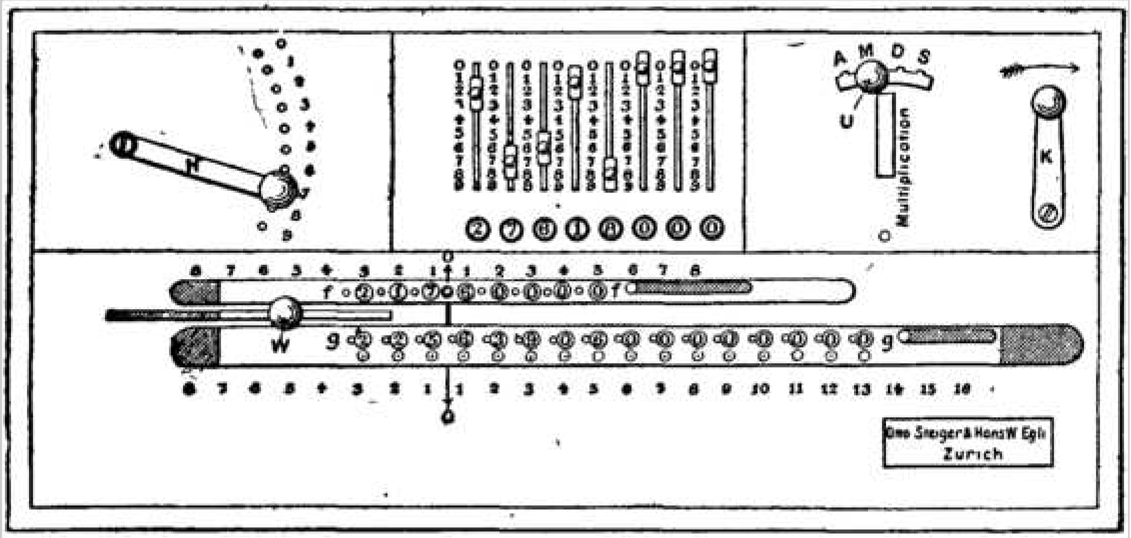

FIG. 3.

Multiplication machines.

The *Steiger-Egli* machine is a multiplication machine, of which fig. 3 gives a picture as it appears to the manipulator. The lower part of the figure contains, under the covering plate, a carriage with two rows of windows for the figures marked *ff* and *gg*. On pressing down the button W the carriage can be moved to right or left. Under each window is a figure disk, as in the Thomas machine. The upper part has three sections. The one to the right contains the handle K for working the machine, and a button U for setting the machine for addition, multiplication, division, or subtraction. In the middle section a number of parallel slots are seen, with indices which can each be set to one of the numbers 0 to 9. Below each slot, and parallel to it, lies a shaft of square section on which a toothed wheel, the A-wheel, slides to and fro with the index in the slot. Below these wheels again lie 9 toothed racks at right angles to the slots. By setting the index in any slot the wheel below it comes into gear with one of these racks. On moving the rack, the wheels turn their shafts and the figure disks gg opposite to them. The dimensions are such that a motion of a rack through 1 cm. turns the figure disk through one "step" or adds 1 to the figure under the window. The racks are moved by an arrangement contained in the section to the left of the slots. There is a vertical plate called the multiplication table block, or more shortly, the *block*. From it project rows of horizontal rods of lengths varying from 0 to 9 centimetres. If one of these rows is brought opposite the row of racks and then pushed forward to the right through 9 cm., each rack will move and add to its figure disk a number of units equal to the number of centimetres of the rod which operates on it. The block has a square face divided into a hundred squares. Looking at its face from the right—*i.e.* from the side where the racks lie—suppose the horizontal rows of these squares numbered from 0 to 9, beginning at the top, and the columns numbered similarly, the 0 being to the right; then the multiplication table for numbers 0 to 9 can be placed on these squares. The row 7 will therefore contain the numbers 63, 56, ... 7, 0. Instead of these numbers, each square receives two "rods" perpendicular to the plate, which may be called the units-rod and the tens-rod. Instead of the number 63 we have thus a tens-rod 6 cm. and a units-rod 3 cm. long. By aid of a lever H the block can be raised or lowered so that any row of the block comes to the level of the racks, the units-rods being opposite the ends of the racks.

The action of the machine will be understood by considering an example. Let it be required to form the product 7 times 385. The indices of three consecutive slots are set to the numbers 3, 8, 5 respectively. Let the windows *gg* opposite these slots be called *a*, *b*, *c*. Then to the figures shown at these windows we have to add 21, 56, 35 respectively. This is the same thing as adding first the number 165, formed by the units of each place, and next 2530 corresponding to the tens; or again, as adding first 165, and then moving the carriage one step to the right, and adding 253. The first is done by moving the block with the units-rods opposite the racks forward. The racks are then put out of gear, and together with the block brought back to their normal position; the block is moved sideways to bring the tens-rods opposite the racks, and again moved forward, adding the tens, the carriage having also been moved forward as required. This complicated movement, together with the necessary carrying, is actually performed by one turn of the handle. During the first quarter-turn the block moves forward, the units-rods coming into operation. During the second quarter-turn the carriage is put out of gear, and moved one step to the right while the necessary carrying is performed; at the same time the block and the racks are moved back, and the block is shifted so as to bring the tens-rods opposite the racks. During the next two quarter-turns the process is repeated, the block ultimately returning to its original position. Multiplication by a number with more places is performed as in the Thomas. The advantage of this machine over the Thomas in saving time is obvious. Multiplying by 817 requires in the Thomas 16 turns of the handle, but in the Steiger-Egli only 3 turns, with 3 settings of the lever H. If the lever H is set to 1 we have a simple addition machine like the Thomas or the Brunsviga. The inventors state that the product of two 8-figure numbers can be got in 6-7 seconds, the quotient of a 6-figure number by one of 3 figures in the same time, while the square root to 5 places of a 9-figure number requires 18 seconds.

Machines of far greater powers than the arithmometers mentioned have been invented by Babbage and by Scheutz. A description is impossible without elaborate drawings. The following account will afford some idea of the working of Babbage's difference machine. Imagine a number of striking clocks placed in a row, each with only an hour hand, and with only the striking apparatus retained. Let the hand of the first clock be turned. As it comes opposite a number on the dial the clock strikes that number of times. Let this clock be connected with the second in such a manner that by each stroke of the first the hand of the second is moved from one number to the next, but can only strike when the first comes to rest. If the second hand stands at 5 and the first strikes 3, then when this is done the second will strike 8; the second will act similarly on the third, and so on. Let there be four such clocks with hands set to the numbers 6, 6, 1, 0 respectively. Now set the third clock striking 1, this sets the hand of the fourth clock to 1; strike the second (6), this puts the third to 7 and the fourth to 8. Next strike the first (6); this moves the other hands to 12, 19, 27 respectively, and now repeat the striking of the first. The hand of the fourth clock will then give in succession the numbers 1, 8, 27, 64, &c., being the cubes of the natural numbers. The numbers thus obtained on the last dial will have the differences given by those shown in succession on the dial before it, their differences by the next, and so on till we come to the constant difference on the first dial. A function

$$y = a + bx + cx^2 + dx^3 + ex^4$$

gives, on increasing x always by unity, a set of values for which the fourth difference is constant. We can, by an arrangement like the above, with five clocks calculate y for $x = 1, 2, 3, \ldots$ to any extent. This is the principle of Babbage's difference machine. The clock dials have to be replaced by a series of dials as in the arithmometers described, and an arrangement has to be made to drive the whole by turning one handle by hand or some other power. Imagine further that with the last clock is connected a kind of typewriter which prints the number, or, better, impresses the number in a soft substance from which a stereotype casting can be taken, and we have a machine which, when once set for a given formula like the above, will automatically print, or prepare stereotype plates for the printing of, tables of the function without any copying or typesetting, thus excluding all possibility of errors. Of this "Difference engine," as Babbage called it, a part was finished in 1834, the government having contributed £17,000 towards the cost. This great expense was chiefly due to the want of proper machine tools.

Meanwhile Babbage had conceived the idea of a much more powerful machine, the "analytical engine," intended to perform any series of possible arithmetical operations. Each of these was to be communicated to the machine by aid of cards with holes punched in them into which levers could drop. It was long taken for granted that Babbage left complete plans; the committee of the British Association appointed to consider this question came, however, to the conclusion (*Brit. Assoc. Report*, 1878, pp. 92-102) that no detailed working drawings existed at all; that the drawings left were only diagrammatic and not nearly sufficient to put into the hands of a draughtsman for making working plans; and "that in the present state of the design it is not more than a theoretical possibility." A full account of the work done by Babbage in connexion with calculating machines, and much else published by others in connexion therewith, is contained in a work published by his son, General Babbage.

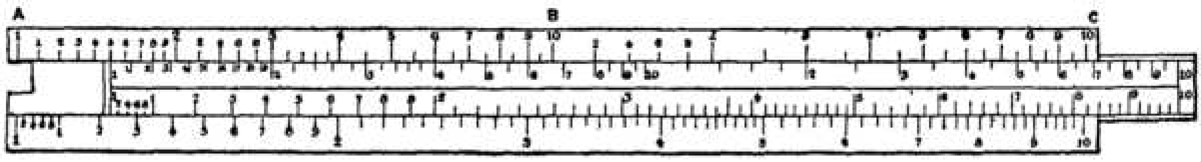

FIG. 4.

Slide rules.

Slide rules are instruments for performing logarithmic calculations mechanically, and are extensively used, especially where only rough approximations are required. They are almost as old as logarithms themselves. Edmund Gunter drew a "logarithmic line" on his "Scales" as follows (fig. 4):—On a line AB lengths are set off to scale to represent the common logarithms of the numbers 1 2 3 ... 10, and the points thus obtained are marked with these numbers. As log 1 = 0, the beginning A has the number 1 and B the number 10, hence the unit of length is AB, as log 10 = 1. The same division is repeated from B to C. The distance 1,2 thus represents log 2, 1,3 gives log 3, the distance between 4 and 5 gives log 5 - log 4 = log 5/4, and so for others. In order to multiply two numbers, say 2 and 3, we have log 2 × 3 = log 2 + log 3. Hence, setting off the distance 1,2 from 3 forward by the aid of a pair of compasses will give the distance log 2 + log 3, and will bring us to 6 as the required product. Again, if it is required to find 4/5 of 7, set off the distance between 4 and 5 from 7 backwards, and the required number will be obtained. In the actual scales the spaces between the numbers are subdivided into 10 or even more parts, so that from two to three figures may be read. The numbers 2, 3 ... in the interval BC give the logarithms of 10 times the same numbers in the interval AB; hence, if the 2 in the latter means 2 or .2, then the 2 in the former means 20 or 2.

Soon after Gunter's publication (1620) of these "logarithmic lines," Edmund Wingate (1672) constructed the slide rule by repeating the logarithmic scale on a tongue or "slide," which could be moved along the first scale, thus avoiding the use of a pair of compasses. A clear idea of this device can be formed if the scale in fig. 4 be copied on the edge of a strip of paper placed against the line A C. If this is now moved to the right till its 1 comes opposite the 2 on the first scale, then the 3 of the second will be opposite 6 on the top scale, this being the product of 2 and 3; and in this position every number on the top scale will be twice that on the lower. For every position of the lower scale the ratio of the numbers on the two scales which coincide will be the same. Therefore multiplications, divisions, and simple proportions can be solved at once.

Dr John Perry added log log scales to the ordinary slide rule in order to facilitate the calculation of a^x or e^x according to the formula $\log \log a^x = \log \log a + \log x$. These rules are manufactured by A.G. Thornton of Manchester.

Many different forms of slide rules are now on the market. The handiest for general use is the Gravet rule made by Tavernier-Gravet in Paris, according to instructions of the mathematician V.M.A. Mannheim of the École Polytechnique in Paris. It contains at the back of the slide scales for the logarithms of sines and tangents so arranged that they can be worked with the scale on the front. An improved form is now made by Davis and Son of Derby, who engrave the scales on white celluloid instead of on box-wood, thus greatly facilitating the readings. These scales have the distance from one to ten about twice that in fig. 4. Tavernier-Gravet makes them of that size and longer, even ½ metre long. But they then become somewhat unwieldy, though they allow of reading to more figures. To get a handy long scale Professor G. Fuller has constructed a spiral slide rule drawn on a cylinder, which admits of reading to three and four figures. The handiest of all is perhaps the "Calculating Circle" by Boucher, made in the form of a watch. For various purposes special adaptations of the slide rules are met with—for instance, in various exposure meters for photographic purposes. General Strachey introduced slide rules into the Meteorological Office for performing special calculations. At some blast furnaces a slide rule has been used for determining the amount of coke and flux required for any weight of ore. Near the balance a large logarithmic scale is fixed with a slide which has three indices only. A load of ore is put on the scales, and the first index of the slide is put to the number giving the weight, when the second and third point to the weights of coke and flux required.

By placing a number of slides side by side, drawn if need be to different scales of length, more complicated calculations may be performed. It is then convenient to make the scales circular. A number of rings or disks are mounted side by side on a cylinder, each having on its rim a log-scale.

The "Callendar Cable Calculator," invented by Harold Hastings and manufactured by Robert W. Paul, is of this kind. In it a number of disks are mounted on a common shaft, on which each turns freely unless a button is pressed down whereby the disk is clamped to the shaft. Another disk is fixed to the shaft. In front of the disks lies a fixed zero line. Let all disks be set to zero and the shaft be turned, with the first disk clamped, till a desired number appears on the zero line; let then the first disk be released and the second clamped and so on; then the fixed disk will add up all the turnings and thus give the product of the numbers shown on the several disks. If the division on the disks is drawn to different scales, more or less complicated calculations may be rapidly performed. Thus if for some purpose the value of say $ab^3 \sqrt{c}$ is required for many different values of a, b, c, three movable disks would be needed with divisions drawn to scales of lengths in the proportion 1: 3: ½. The instrument now on sale contains six movable disks.

Continuous Calculating Machines or Integrators.—In order to measure the length of a curve, such as the road on a map, a wheel is rolled along it. For one revolution of the wheel the path described by its point of contact is equal to the circumference of the wheel. Thus, if a cyclist counts the number of revolutions of his front wheel he can calculate the distance ridden by multiplying that number by the circumference of the wheel. An ordinary cyclometer is nothing but an arrangement for counting these revolutions, but it is graduated in such a manner that it gives at once the distance in miles. On the same principle depend a number of instruments which, under various fancy names, serve to measure the length of any curve; they are in the shape of a small meter chiefly for the use of cyclists. They all have a small wheel which is rolled along the curve to be measured, and this sets a hand in motion which gives the reading on a dial. Their accuracy is not very great, because it is difficult to place the wheel so on the paper that the point of contact lies exactly over a given point; the beginning and end of the readings are therefore badly defined. Besides, it is not easy to guide the wheel along the curve to which it should always lie tangentially. To obviate this defect more complicated curvometers or kartometers have been devised. The handiest seems to be that of G. Coradi. He uses two wheels; the tracing-point, halfway between them, is guided along the curve, the line joining the wheels being kept normal to the curve. This is pretty easily done by eye; a constant deviation of 8° from this direction produces an error of only 1%. The sum of the two readings gives the length. E. Fleischhauer uses three, five or more wheels arranged symmetrically round a tracer whose point is guided along the curve; the planes of the wheels all pass through the tracer, and the wheels can only turn in one direction. The sum of the readings of all the wheels gives approximately the length of the curve, the approximation increasing with the number of the wheels used. It is stated that with three wheels practically useful results can be obtained, although in this case the error, if the instrument is consistently handled so as always to produce the greatest inaccuracy, may be as much as 5%.

Curvometers.

Planimeters are instruments for the determination by mechanical means of the area of any figure. A pointer, generally called the "tracer," is guided round the boundary of the figure, and then the area is read off on the recording apparatus of the instrument. The simplest and most useful is Amsler's (fig. 5). It consists of two bars of metal OQ and QT, which are hinged together at Q. At O is a needle-point which is driven into the drawing-board, and at T is the tracer. As this is guided round the boundary of the figure a wheel W mounted on QT rolls on the paper, and the turning of this wheel measures, to some known scale, the area. We shall give the theory of this instrument fully in an elementary manner by aid of geometry. The theory of other planimeters can then be easily understood.

Planimeters.

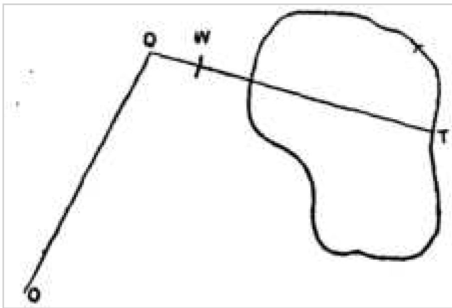

FIG. 5.

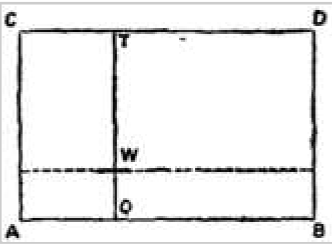

FIG. 6.

Consider the rod QT with the wheel W, without the arm OQ. Let it be placed with the wheel on the paper, and now moved perpendicular to itself from AC to BD (fig. 6). The rod sweeps over, or generates, the area of the rectangle ACDB = lp, where l denotes the length of the rod and p the distance AB through which it has been moved. This distance, as measured by the rolling of the wheel, which acts as a curvometer, will be called the "roll" of the wheel and be denoted by w. In this case $p = w$, and the area P is given by $P = wl$. Let the circumference of the wheel be divided into say a hundred equal parts u; then w registers the number of *u's* rolled over, and w therefore gives the number of areas lu contained in the rectangle. By suitably selecting the radius of the wheel and the length l, this area lu may be any convenient unit, say a square inch or square centimetre. By changing l the unit will be changed.

Again, suppose the rod to turn (fig. 7) about the end Q, then it will describe an arc of a circle, and the rod will generate an area $\frac{1}{2}l^2\theta$, where θ is the angle AQB through which the rod has turned. The wheel will roll over an arc $c\theta$, where c is the distance of the wheel from Q. The "roll" is now $w = c\theta$; hence the area generated is

$$P = \tfrac{1}{2}\, l^2/c\; w,$$

and is again determined by w.

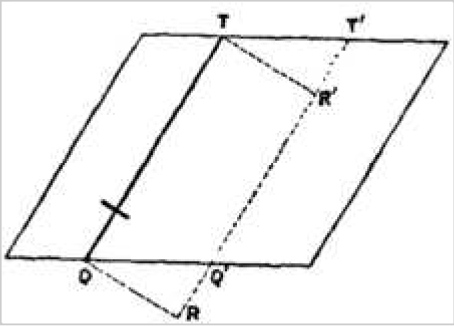

FIG. 8.

Next let the rod be moved parallel to itself, but in a direction not perpendicular to itself (fig. 8). The wheel will now not simply roll. Consider a *small* motion of the rod from QT to Q′T′. This may be resolved into the motion to RR′ perpendicular to the rod, whereby the rectangle QTR′R is generated, and the sliding of the rod along itself from RR′ to Q′T′. During this second step no area will be generated.

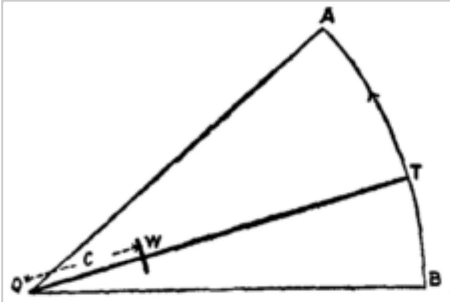

FIG. 7.

During the first step the roll of the wheel will be QR, whilst during the second step there will be no roll at all. The roll of the wheel will therefore measure the area of the rectangle which equals the parallelogram QTT′Q′. If the whole motion of the rod be considered as made up of a very great number of small steps, each resolved as stated, it will be seen that the roll again measures the area generated. But it has to be noticed that now the wheel does not only roll, but also slips, over the paper. This, as will be pointed out later, may introduce an error in the reading.

We can now investigate the most general motion of the rod. We again resolve the motion into a number of small steps. Let (fig. 9) AB be one position, CD the next after a step so small that the arcs AC and BD over which the ends have passed may be considered as straight lines. The area generated is ABDC. This motion we resolve into a step from AB to CB′, parallel to AB and a turning about C from CB′ to CD, steps such as have been investigated. During the first, the "roll" will be p the altitude of the parallelogram; during the second will be $c\theta$. Therefore

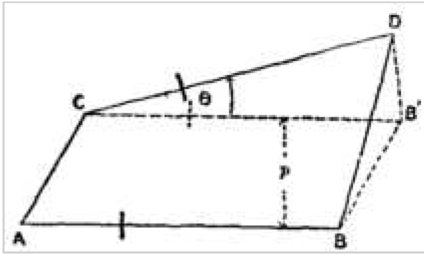

FIG. 9.

$$w = p + c\theta.$$

The area generated is $lp + \frac{1}{2} l^2\theta$, or, expressing p in terms of w, $lw + (\frac{1}{2}l^2 - lc)\theta$. For a finite motion we get the area equal to the sum of the areas generated during the different steps. But the wheel will continue rolling, and give the whole roll as the sum of the rolls for the successive steps. Let then w denote the whole roll (in fig. 10), and let α denote the sum of all the small turnings θ; then the area is

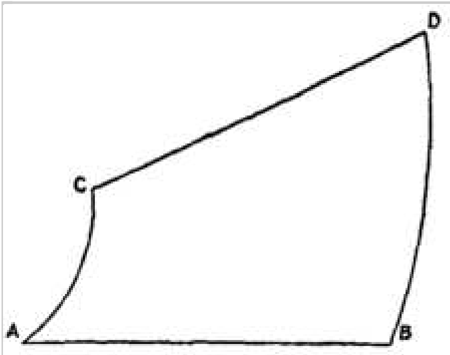

FIG. 10.

$$P = lw + (\tfrac{1}{2}l^2 - lc)\alpha \ . \ . \ . \ (1)$$

Here α is the angle which the last position of the rod makes with the first. In all applications of the planimeter the rod is brought back to its original position. Then the angle α is either zero, or it is 2π if the rod has been once turned quite round.

Hence in the first case we have

$$P = lw \ . \ . \ . \ (2a)$$

and w gives the area as in case of a rectangle.

In the other case

$$P = lw + lC \ . \ . \ . \ (2b)$$

where $C = (\tfrac{1}{2}l - c)2\pi$, if the rod has once turned round. The number C will be seen to be always the same, as it depends only on the dimensions of the instrument. Hence now again the area is determined by w if C is known.

Thus it is seen that the area generated by the motion of the rod can be measured by the roll of the wheel; it remains to show how any given area can be generated by the rod. Let the rod move in any manner but return to its original position. Q and T then describe closed curves. Such motion may be called cyclical. Here the theorem holds: —*If a rod QT performs a cyclical motion, then the area generated equals the difference of the areas enclosed by the paths of T and Q respectively.* The truth of this proposition will be seen from a figure. In fig. 11 different positions of the moving rod QT have been marked, and its motion can be easily followed. It will be seen that every part of the area TT′BB′ will be passed over once and always by a *forward motion* of the rod, whereby the wheel will *increase* its roll. The area AA′ QQ′ will also be swept over once, but with a *backward* roll; it must therefore be counted as negative. The area between the curves is passed over twice, once with a forward and once with a backward roll; it therefore counts once positive and once

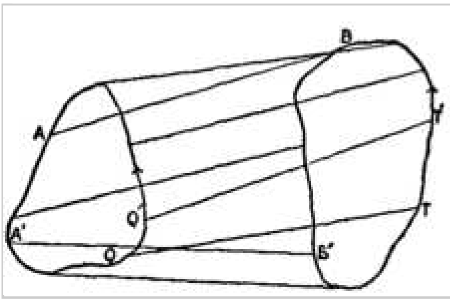

FIG. 11.

negative; hence not at all. In more complicated figures it may happen that the area within one of the curves, say TT′BB′, is passed over several times, but then it will be passed over once more in the forward direction than in the backward one, and thus the theorem will still hold.

To use Amsler's planimeter, place the pole O on the paper *outside* the figure to be measured. Then the area generated by QT is that of the figure, because the point Q moves on an arc of a circle to and fro enclosing no area. At the same time the rod comes back without making a complete rotation. We have therefore in formula (1), $\alpha = 0$; and hence

$$P = lw,$$

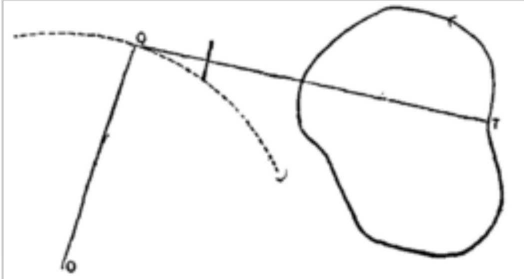

FIG. 12.

which is read off. But if the area is too large the pole O may be placed within the area. The rod describes the area between the boundary of the figure and the circle with radius r = OQ, whilst the rod turns once completely round, making $\alpha = 2\pi$. The area measured by the wheel is by formula (1), $lw + (\tfrac{1}{2}l^2 - lc)\, 2\pi$.

To this the area of the circle πr^2 must be added, so that now

$$P = lw + (\tfrac{1}{2}l^2 - lc)2\pi + \pi r^2,$$

or

$$P = lw + C,$$

where

$$C = (\tfrac{1}{2}l^2 - lc)2\pi + \pi r^2,$$

is a constant, as it depends on the dimensions of the instrument alone. This constant is given with each instrument.

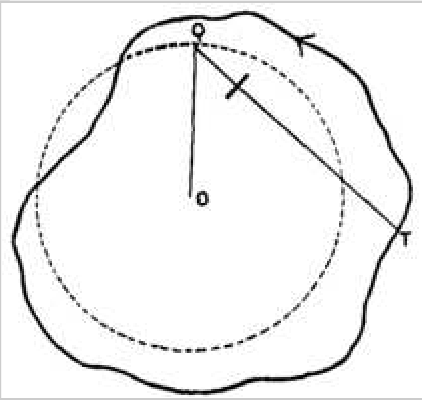

FIG. 13.

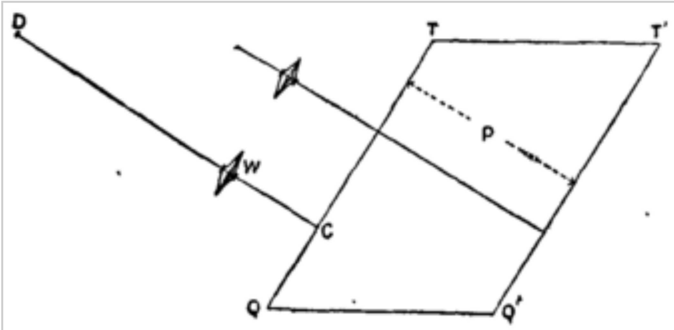

FIG. 14.

Amsler's planimeters are made either with a rod QT of fixed length, which gives the area therefore in terms of a fixed unit, say in square inches, or else the rod can be moved in a sleeve to which the arm OQ is hinged (fig. 13). This makes it possible to change the unit lu, which is proportional to l.

In the planimeters described the recording or integrating apparatus is a smooth wheel rolling on the paper or on some other surface. Amsler has described another recorder, viz. a wheel with a sharp edge. This will roll on the paper but not slip. Let the rod QT carry with it an arm CD perpendicular to it. Let there be mounted on it a wheel W, which can slip along and turn about it. If now QT is moved parallel to itself to Q′T′, then W will roll without slipping parallel to QT, and slip along CD. This amount of slipping will equal the perpendicular distance between QT and Q′T′, and therefore serve to measure the area swept over like the wheel in the machine already described. The turning of the rod will also produce slipping of the wheel, but it will be seen without difficulty that this will cancel during a cyclical motion of the rod, provided the rod does not perform a whole rotation.

Early forms.

The first planimeter was made on the following principles:—A frame FF (fig. 15) can move parallel to OX. It carries a rod TT movable along its own length, hence the tracer T can be guided along any curve ATB. When the rod has been pushed back to Q′Q, the tracer moves along the axis OX. On the frame a cone VCC′ is mounted with its axis sloping so that its top edge is horizontal and parallel to TT′, whilst its vertex V is opposite Q′. As the frame moves it turns the cone. A wheel W is mounted on the rod at T′, or on an axis parallel to and rigidly connected with it. This wheel rests on the top edge of the cone. If now the tracer T, when pulled out through a distance y above Q, be moved parallel to OX through a distance dx, the frame moves through an equal distance, and the cone turns through an angle $d\theta$ proportional to dx. The wheel W rolls on the cone to an amount again proportional to dx, and also proportional to y, its distance from V. Hence the roll of the wheel is proportional to the area ydx described by the rod QT. As T is moved from A to B along the curve the roll of the wheel will therefore be proportional to the area AA′B′B. If the curve is closed, and the tracer moved round it, the roll will measure the area independent of the position of the axis OX, as will be seen by drawing a figure. The cone may with advantage be replaced by a horizontal disk, with its centre at V; this allows of y being negative. It may be noticed at once that the roll of the wheel gives at every moment the area A′ATQ. It will therefore allow of registering a set of values of $\int_a^x y\,dx$ for any values of x, and thus of tabulating the values of any indefinite integral. In this it differs from Amsler's planimeter. Planimeters of this type were first invented in 1814 by the Bavarian engineer Hermann, who, however, published nothing. They were reinvented by Prof. Tito Gonnella of Florence in 1824, and by the Swiss engineer Oppikofer, and improved by Ernst in Paris, the astronomer Hansen in Gotha, and others (see Henrici, *British Association Report*, 1894). But all were driven out of the field by Amsler's simpler planimeter.

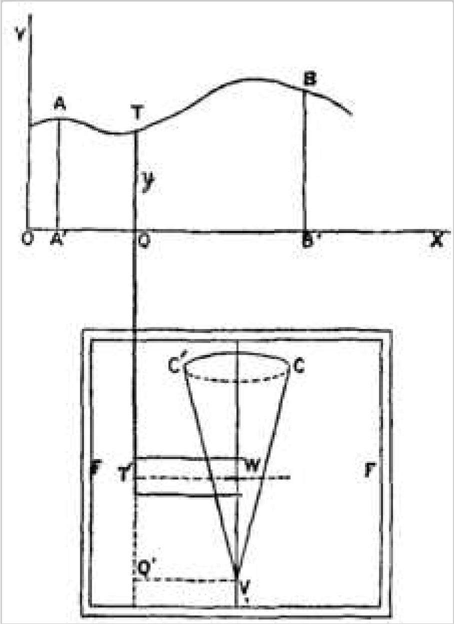

FIG. 15.

FIG. 16.

Hatchet planimeters.

Altogether different from the planimeters described is the hatchet planimeter, invented by Captain Prytz, a Dane, and made by Herr Cornelius Knudson in Copenhagen. It consists of a single rigid piece like fig. 16. The one end T is the tracer, the other Q has a sharp hatchet-like edge. If this is placed with QT on the paper and T is moved along any curve, Q will follow, describing a "curve of pursuit." In consequence of the sharp edge, Q can only move in the direction of QT, but the whole can turn about Q. Any small step forward can therefore be considered as made up of a motion along QT, together with a turning about Q. The latter motion alone generates an area. If therefore a line OA = QT is turning about a fixed point O, always keeping parallel to QT, it will sweep over an area equal to that generated by the more general motion of QT. Let now (fig. 17) QT be placed on OA, and T be guided round the closed curve in the sense of the arrow. Q will describe a curve OSB. It may be made visible by putting a piece of "copying paper" under the hatchet. When T has returned to A the hatchet has the position BA. A line turning from OA about O kept parallel to QT will describe the circular sector OAC, which is equal in magnitude and sense to AOB. This therefore measures the area generated by the motion of QT. To make this motion cyclical, suppose the hatchet turned about A till Q comes from B to O. Hereby the sector AOB is again described, and again in the positive sense, if it is remembered that it turns about the tracer T fixed at A. The whole area now generated is therefore twice the area of this sector, or equal to OA. OB, where OB is measured along the arc. According to the theorem given above, this area also equals the area of the given curve less the area OSBO. To make this area disappear, a slight modification of the motion of QT is required. Let the tracer T be moved, both from the first position OA and the last BA of the rod, along some straight line AX. Q describes curves OF and BH respectively. Now begin the motion with T at some point R on AX, and move it along this line to A, round the curve and back to R. Q will describe the curve DOSBED, if the motion is again made cyclical by turning QT with T fixed at A. If R is properly selected, the path of Q will cut itself, and parts of the area will be positive, parts negative, as marked in the figure, and may therefore be made to vanish. When this is done the area of the curve will equal twice the area of the sector RDE. It is therefore equal to the arc DE multiplied by the length QT; if the latter equals 10 in., then 10 times the number of inches contained in the arc DE gives the number of square inches contained within the given figure. If the area is not too large, the arc DE may be replaced by the straight line DE.

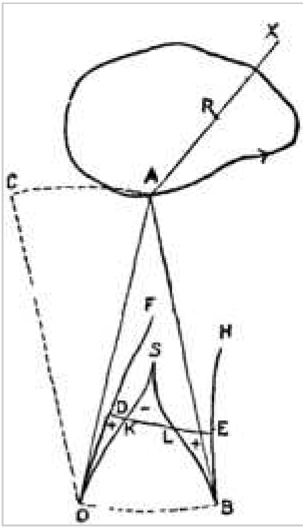

FIG. 17.

To use this simple instrument as a planimeter requires the possibility of selecting the point R. The geometrical theory here given has so far failed to give any rule. In fact, every line through any point in the curve contains such a point. The analytical theory of the inventor, which is very similar to that given by F.W. Hill (*Phil. Mag.* 1894), is too complicated to repeat here. The integrals expressing the area generated by QT have to be expanded in a series. By retaining only the most important terms a result is obtained which comes to this, that if the mass-centre of the area be taken as R, then A may be any point on the curve. This is only approximate. Captain Prytz gives the following instructions:—Take a point R as near as you can guess to the mass-centre, put the tracer T on it, the knife-edge Q outside; make a mark on the paper by pressing the knife-edge into it; guide the tracer from R along a straight line to a point A on the boundary, round the boundary, and back from A to R; lastly, make again a mark with the knife-edge, and measure the distance c between the marks; then the area is nearly cl, where l = QT. A nearer approximation is obtained by repeating the operation after turning QT through 180° from the original position, and using the mean of the two values of c thus obtained. The greatest dimension of the area should not exceed ½l, otherwise the area must be divided into parts which are determined separately. This condition being fulfilled, the instrument gives very satisfactory results, especially if the figures to be measured, as in the case of indicator diagrams, are much of the same shape, for in this case the operator soon learns where to put the point R.

Integrators.

Integrators serve to evaluate a definite integral $\int_a^b f(x)\,dx$. If we plot out the curve whose equation is $y = f(x)$, the integral $\int y dx$ between the proper limits represents the area of a figure bounded by the curve, the axis of x, and the ordinates at $x=a$, $x=b$. Hence if the curve is drawn, any planimeter may be used for finding the value of the integral. In this sense planimeters are integrators. In fact, a planimeter may often be used with advantage to solve problems more complicated than the determination of a mere area, by converting the one problem graphically into the other. We give an example:—

Let the problem be to determine for the figure ABG (fig. 18), not only the area, but also the first and second moment with regard to the axis XX. At a distance a draw a line, C′D′, parallel to XX. In the figure draw a number of lines parallel to AB. Let CD be one of them. Draw C and D vertically upwards to C′D′, join these points to some point O in XX, and mark the points C_1D_1 where OC′ and OD′ cut CD. Do this for a sufficient number of lines, and join the points C_1D_1 thus obtained. This gives a new curve, which may be called the first derived curve. By the same process get a new curve from this, the second derived curve. By aid of a planimeter determine the areas P, P_1, P_2, of these three curves. Then, if $\bar{x}$ is the distance of the mass-centre of the given area from XX; $\bar{x}_1$ the same quantity for the first derived figure, and $I = Ak^2$ the moment of inertia of the first figure, k its radius of gyration, with regard to XX as axis, the following relations are easily proved:—

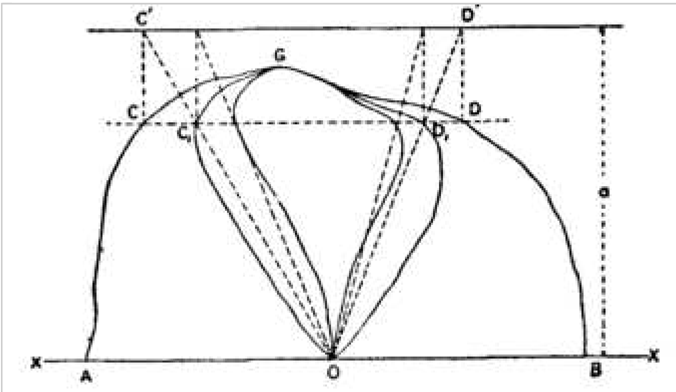

FIG. 18.

$$P\bar{x} = aP_1;\ P_1\bar{x}_1 = aP_2;\ I = aP_1\bar{x}_1 = a^2P_1P_2;\ k^2 = \overline{xx_1},$$

which determine P, $\bar{x}$ and I or k. Amsler has constructed an integrator which serves to determine these quantities by guiding a tracer once round the boundary of the given figure (see below). Again, it may be required to find the value of an integral $\int y\varphi(x)dx$ between given limits where $\varphi(x)$ is a simple function like sin nx, and where y is given as the ordinate of a curve. The harmonic analysers described below are examples of instruments for evaluating such integrals.

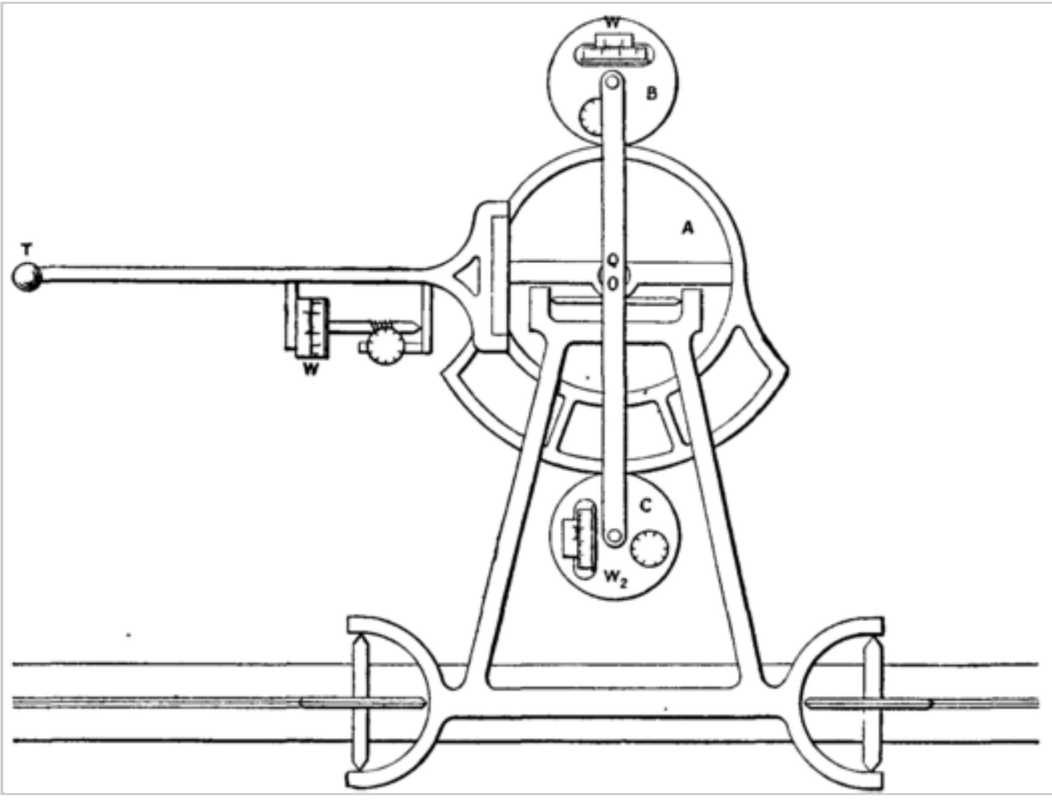

FIG. 19.

Amsler has modified his planimeter in such a manner that instead of the area it gives the first or second moment of a figure about an axis in its plane. An instrument giving all three quantities simultaneously is known as Amsler's integrator or moment-planimeter. It has one tracer, but three recording wheels. It is mounted on a carriage which runs on a straight rail (fig. 19). This carries ***Amsler's Integrator.*** a horizontal disk A, movable about a vertical axis Q. Slightly more than half the circumference is circular with radius $2a$, the other part with radius $3a$. Against these gear two disks, B and C, with radii a; their axes are fixed in the carriage. From the disk A extends to the left a rod OT of length l, on which a recording wheel W is mounted. The disks B and C have also recording wheels, W_1 and W_2, the axis of W_1 being perpendicular, that of W_2 parallel to OT. If now T is guided round a figure F, O will move to and fro in a straight line. This part is therefore a simple planimeter, in which the one end of the arm moves in a straight line instead of in a circular arc. Consequently, the "roll" of W will record the area of the figure. Imagine now that the disks B and C also receive arms of length l from the centres of the disks to points T_1 and T_2, and in the direction of the axes of the wheels. Then these arms with their wheels will again be planimeters. As T is guided round the given figure F, these points T_1 and T_2 will describe closed curves, F_1 and F_2, and the "rolls" of W_1 and W_2 will give their areas A_1 and A_2. Let XX (fig. 20) denote the line, parallel to the rail, on which O moves; then when T lies on this line, the arm BT_1 is perpendicular to XX, and CT_2 parallel to it. If OT is turned through an angle θ, clockwise, BT_1 will turn counter-clockwise through an angle 2θ, and CT_2 through an angle 3θ, also counter-clockwise. If in this position T is moved through a distance x parallel to the axis XX, the points T_1 and T_2 will move parallel to it through an equal distance. If now the first arm is turned through a small angle $d\theta$, moved back through a distance x, and lastly turned back through the angle $d\theta$, the tracer T will have described the boundary of a small strip of area. We divide the given figure into such strips. Then to every such strip will correspond a strip of equal length x of the figures described by T_1 and T_2.

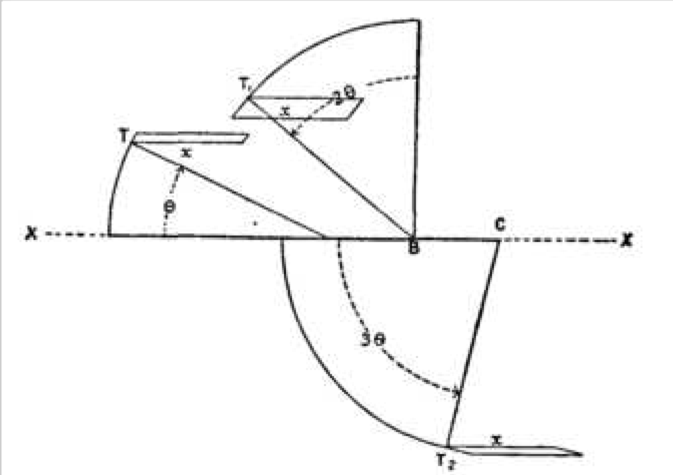

FIG. 20.

The distances of the points, T, T_1, T_2, from the axis XX may be called y, y_1, y_2. They have the values

$$y = l \sin \theta,\ y_1 = l \cos 2\theta,\ y_2 = -l \sin 3\theta,$$

from which

$$dy = l \cos \theta . d\theta,\ dy_1 = -\ 2l \sin 2\theta . d\theta,\ dy_2 = -\ 3l \cos 3\theta . d\theta.$$

The areas of the three strips are respectively

$$dA = xdy,\ dA_1 = xdy_1,\ dA_2 = xdy_2.$$

Now dy_1 can be written $dy_1 = -\ 4l \sin \theta \cos \theta d\theta = -\ 4 \sin \theta dy$; therefore

$$dA_1 = -\ 4 \sin \theta . dA = -\ \tfrac{4}{l} y dA;$$

whence

$$A_1 = -\ \tfrac{4}{l} \int y dA = -\ \tfrac{4}{l} A\bar{y},$$

where A is the area of the given figure, and $\bar{y}$ the distance of its mass-centre from the axis XX. But A_1 is the area of the second figure F_1, which is proportional to the reading of W_1. Hence we may say

$$A\bar{y} = C_1 w_1,$$

where C_1 is a constant depending on the dimensions of the instrument. The negative sign in the expression for A_1 is got rid of by numbering the wheel W_1 the other way round.

Again

$$dy_2 = -\ 3l \cos \theta \{4 \cos^2 \theta - 3\}\ d\theta = -\ 3\ \{4 \cos^2 \theta - 3\}\ dy = -\ 3\ \{\ \tfrac{4}{l^2} y^2 - 3\}\ dy,$$

which gives

$$dA_2 = -\ \tfrac{12}{l^2} y^2 dA + 9dA,$$

and

$A_2 = - \frac{12}{l^2} \int y^2 dA + 9A.$

But the integral gives the moment of inertia I of the area A about the axis XX. As A_2 is proportional to the roll of W_2, A to that of W, we can write

$I = Cw - C_2 w_2,$
$A\bar{y} = C_1 w_1,$
$A = C_c w.$

If a line be drawn parallel to the axis XX at the distance $\bar{y}$, it will pass through the mass-centre of the given figure. If this represents the section of a beam subject to bending, this line gives for a proper choice of XX the neutral fibre. The moment of inertia for it will be $I + A\bar{y}^2$. Thus the instrument gives at once all those quantities which are required for calculating the strength of the beam under bending. One chief use of this integrator is for the calculation of the displacement and stability of a ship from the drawings of a number of sections. It will be noticed that the length of the figure in the direction of XX is only limited by the length of the rail.

This integrator is also made in a simplified form without the wheel W_2. It then gives the area and first moment of any figure.

Integraphs.

While an integrator determines the value of a definite integral, hence a mere constant, an integraph gives the value of an indefinite integral, which is a function of *x*. Analytically if *y* is a given function *f*(*x*) of *x* and

$Y = \int_c^x y\,dx$ or $Y = \int y dx + \text{const.}$

the function Y has to be determined from the condition

$\frac{dY}{dx} = y.$

Graphically $y = f(x)$ is either given by a curve, or the graph of the equation is drawn: *y*, therefore, and similarly Y, is a length. But $\frac{dY}{dx}$ is in this case a mere number, and cannot equal a length *y*. Hence we introduce an arbitrary constant length *a*, the unit to which the integraph draws the curve, and write

$\frac{dY}{dx} = \frac{y}{a}$ and $aY = \int y dx$

Now for the Y-curve $\frac{dY}{dx} = \tan \varphi$, where φ is the angle between the tangent to the curve, and the axis of *x*. Our condition therefore becomes

$\tan \varphi = \frac{y}{c}.$

This φ is easily constructed for any given point on the y-curve:—From the foot B′ (fig. 21) of the ordinate *y* = B′B set off, as in the figure, B′D = *a*, then angle BDB′ = φ. Let now DB′ with a perpendicular B′B move along the axis of *x*, whilst B follows the *y*-curve, then a pen P on B′B will describe the Y-curve provided it moves at every moment in a direction parallel to BD. The object of the integraph is to draw this new curve when the tracer of the instrument is guided along the *y*-curve.

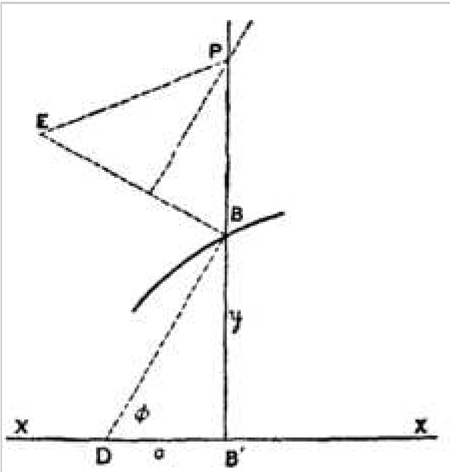

FIG. 21.

The first to describe such instruments was Abdank-Abakanowicz, who in 1889 published a book in which a variety of mechanisms to obtain the object in question are described. Some years later G. Coradi, in Zürich, carried out his ideas. Before this was done, C.V. Boys, without knowing of Abdank-Abakanowicz's work, actually made an integraph which was exhibited at the Physical Society in 1881. Both make use of a sharp edge wheel. Such a wheel will not slip sideways; it will roll forwards along the line in which its plane intersects the plane of the paper, and while rolling will be able to turn gradually about its point of contact. If then the angle between its direction of rolling and the *x*-axis be always equal to φ, the wheel will roll along the Y-curve required. The axis of *x* is fixed only in direction; shifting it parallel to itself adds a constant to Y, and this gives the arbitrary constant of integration.

In fact, if Y shall vanish for *x* = *c*, or if

$Y = \int_c^x y\,dx,$

then the axis of *x* has to be drawn through that point on the *y*-curve which corresponds to *x* = *c*.

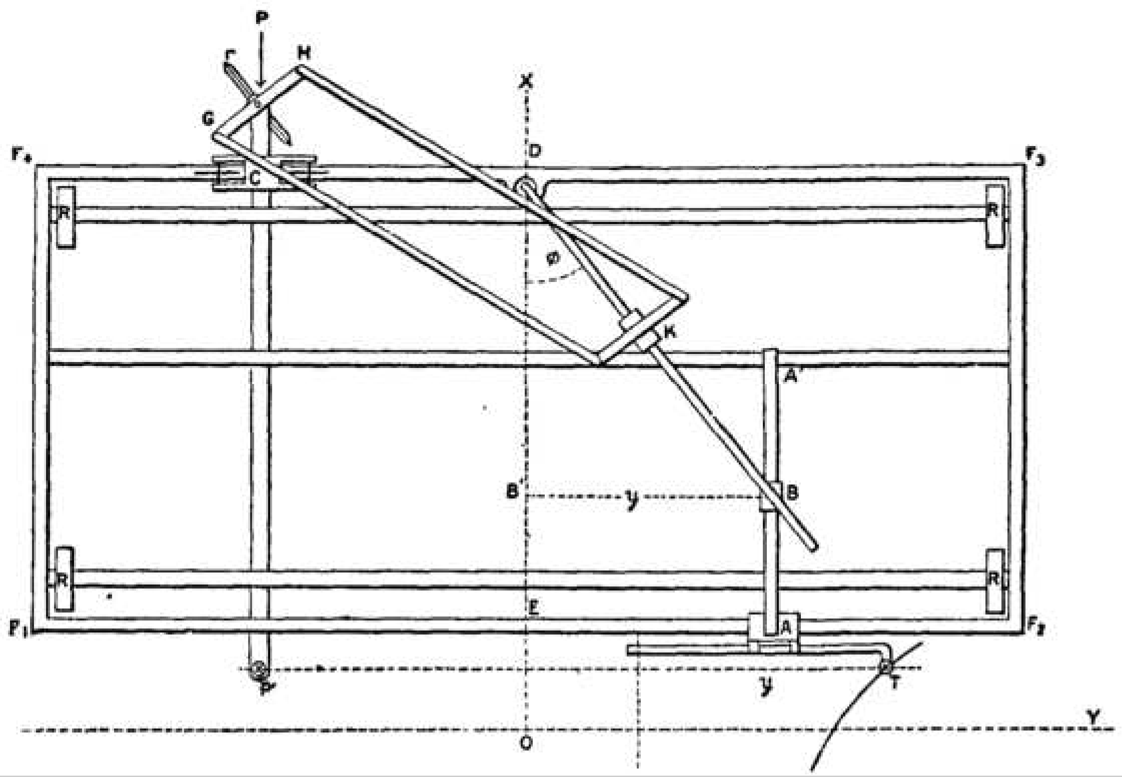

FIG. 22.

In Coradi's integraph a rectangular frame $F_1F_2F_3F_4$ (fig. 22) rests with four rollers R on the drawing board, and can roll freely in the direction OX, which will be called the axis of the instrument. On the front edge F_1F_2 travels a carriage AA′ supported at A′ on another rail. A bar DB can turn about D, fixed to the frame in its axis, and slide through a point B fixed in the carriage AA′. Along it a block K can slide. On the back edge F_3F_4 of the frame another carriage C travels. It holds a vertical spindle with the knife-edge wheel at the bottom. At right angles to the plane of the wheel, the spindle has an arm GH, which is kept parallel to a similar arm attached to K perpendicular to DB. The plane of the knife-edge wheel *r* is therefore always parallel to DB. If now the point B is made to follow a curve whose *y* is measured from OX, we have in the triangle BDB′, with the angle φ at D,

$$\tan \varphi = y/a,$$

where a = DB′ is the constant base to which the instrument works. The point of contact of the wheel *r* or any point of the carriage C will therefore always move in a direction making an angle φ with the axis of *x*, whilst it moves in the *x*-direction through the same distance as the point B on the *y*-curve—that is to say, it will trace out the integral curve required, and so will any point rigidly connected with the carriage C. A pen P attached to this carriage will therefore draw the integral curve. Instead of moving B along the *y*-curve, a tracer T fixed to the carriage A is guided along it. For using the instrument the carriage is placed on the drawing-board with the front edge parallel to the axis of *y*, the carriage A being clamped in the central position with A at E and B at B′ on the axis of *x*. The tracer is then placed on the *x*-axis of the *y*-curve and clamped to the carriage, and the instrument is ready for use. As it is convenient to have the integral curve placed directly opposite to the *y*-curve so that corresponding values of *y* or Y are drawn on the same line, a pen P′ is fixed to C in a line with the tracer.

Boys' integraph was invented during a sleepless night, and during the following days carried out as a working model, which gives highly satisfactory results. It is ingenious in its simplicity, and a direct realization as a mechanism of the principles explained in connexion with fig. 21. The line B′B is represented by the edge of an ordinary T-square sliding against the edge of a drawing-board. The points B and P are connected by two rods BE and EP, jointed at E. At B, E and P are small pulleys of equal diameters. Over these an endless string runs, ensuring that the pulleys at B and P always turn through equal angles. The pulley at B is fixed to a rod which passes through the point D, which itself is fixed in the T-square. The pulley at P carries the knife-edge wheel. If then B and P are kept on the edge of the T-square, and B is guided along the curve, the wheel at P will roll along the Y-curve, it having been originally set parallel to BD. To give the wheel at P sufficient grip on the paper, a small loaded three-wheeled carriage, the knife-edge wheel P being one of its wheels, is added. If a piece of copying paper is inserted between the wheel P and the drawing paper the Y-curve is drawn very sharply.

Integraphs have also been constructed, by aid of which ordinary differential equations, especially linear ones, can be solved, the solution being given as a curve. The first suggestion in this direction was made by Lord Kelvin. So far no really useful instrument has been made, although the ideas seem sufficiently developed to enable a skilful instrument-maker to produce one should there be sufficient demand for it. Sometimes a combination of graphical work with an integraph will serve the purpose. This is the case if the variables are separated, hence if the equation

$$X\,dx + Y\,dy = 0$$

has to be integrated where $X = p(x)$, $Y = \varphi(y)$ are given as curves. If we write

$$au = \int X dx,\ av = \int Y dy,$$

then u as a function of x, and v as a function of y can be graphically found by the integraph. The general solution is then

$$u + v = c$$

with the condition, for the determination for c, that $y = y_0$, for $x = x_0$. This determines $c = u_0 + v_0$, where u_0 and v_0 are known from the graphs of u and v. From this the solution as a curve giving y a function of x can be drawn:—For any x take u from its graph, and find the y for which $v = c - u$, plotting these y against their x gives the curve required. If a periodic function y of x is given by its graph for one period c, it can, according to the theory of Fourier's Series, be expanded in a series.

$$y = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \dots + A_n \cos n\theta + \dots$$

$$+ B_1 \sin\theta + B_2 \sin 2\theta + \dots + B_n \sin n\theta + \dots$$

where $\theta = \frac{2\pi x}{c}$.

The absolute term A_0 equals the mean ordinate of the curve, and can therefore be determined by any planimeter. The other co-efficients are

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta . d\theta;$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \sin n\theta . d\theta.$$

Harmonic analysers.

A harmonic analyser is an instrument which determines these integrals, and is therefore an integrator. The first instrument of this kind is due to Lord Kelvin (*Proc. Roy Soc.*, vol xxiv., 1876). Since then several others have been invented (see Dyck's *Catalogue*; Henrici, *Phil. Mag.*, July 1894; *Phys. Soc.*, 9th March; Sharp, *Phil. Mag.*, July 1894; *Phys. Soc.*, 13th April). In Lord Kelvin's instrument the curve to be analysed is drawn on a cylinder whose circumference equals the period c, and the sine and cosine terms of the integral are introduced by aid of simple harmonic motion. Sommerfeld and Wiechert, of Konigsberg, avoid this motion by turning the cylinder about an axis perpendicular to that of the cylinder. Both these machines are large, and practically fixtures in the room where they are used. The first has done good work in the Meteorological Office in London in the analysis of meteorological curves. Quite different and simpler constructions can be used, if the integrals determining A_n and B_n be integrated by parts. This gives

$$nA_n = -\frac{1}{\pi}\int_0^{2\pi} \sin n\theta . dy;$$

$$nB_n = \frac{1}{\pi}\int_0^{2\pi} \cos n\theta . dy.$$

An analyser presently to be described, based on these forms, has been constructed by Coradi in Zurich (1894). Lastly, a most powerful analyser has been invented by Michelson and Stratton (U.S.A.) (*Phil Mag.*, 1898), which will also be described.

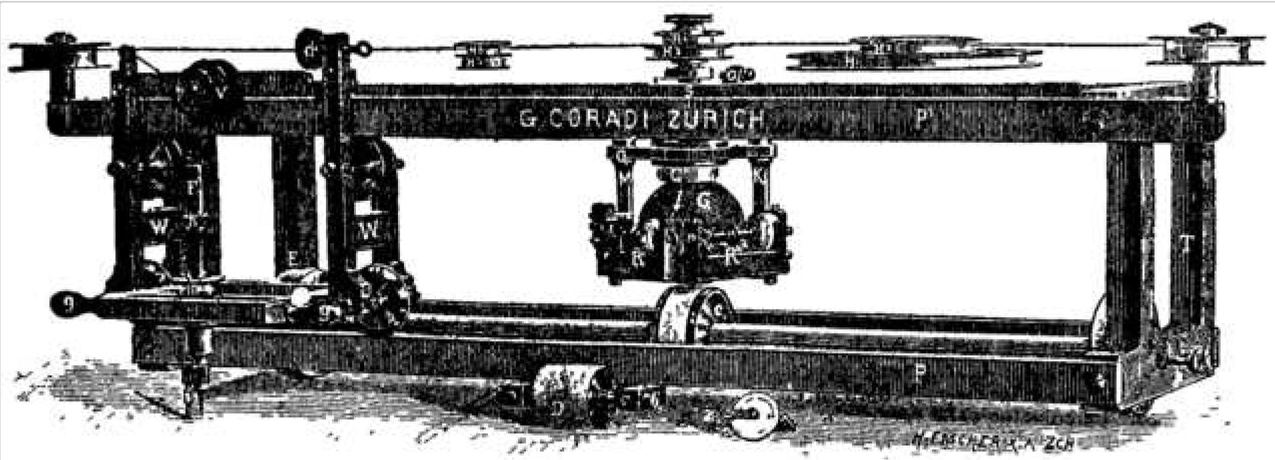

FIG. 23.

The *Henrici-Coradi* analyser has to add up the values of $dy.\sin n\theta$ and $dy.\cos n\theta$. But these are the components of dy in two directions perpendicular to each other, of which one makes an angle $n\theta$ with the axis of x or of θ. This decomposition can be performed by Amsler's registering wheels. Let two of these be mounted, perpendicular to each other, in one horizontal frame which can be turned about a vertical axis, the wheels resting on the paper on which the curve is drawn. When the tracer is placed on the curve at the point $\theta = 0$ the one axis is parallel to the axis of θ. As the tracer follows the curve the frame is made to turn through an angle $n\theta$. At the same time the frame moves with the tracer in the direction of y. For a small motion the two wheels will then register just the components required, and during the continued motion of the tracer along the curve the wheels will add these components, and thus give the values of nA_n and nB_n. The factors $1/\pi$ and $-1/\pi$ are taken account of in the graduation of the wheels. The readings have then to be divided by n to give the coefficients required. Coradi's realization of this idea will be understood from fig. 23. The frame PP′ of the instrument rests on three rollers E, E′, and D. The first two drive an axis with a disk C on it. It is placed parallel to the axis of x of the curve. The tracer is attached to a carriage WW which runs on the rail P. As it follows the curve this carriage moves through a distance x whilst the whole instrument runs forward through a distance y. The wheel C turns through an angle proportional, during each small motion, to dy. On it rests a glass sphere which will therefore also turn about its horizontal axis proportionally, to dy. The registering frame is suspended by aid of a spindle S, having a disk H. It is turned by aid of a wire connected with the carriage WW, and turns n times round as the tracer describes the whole length of the curve. The registering wheels R, R′ rest against the glass sphere and give the values nA_n and nB_n. The value of n can be altered by changing the disk H into one of different diameter. It is also possible to mount on the same frame a number of spindles with registering wheels and glass spheres, each of the latter resting on a separate disk C. As many as five have been introduced. One guiding of the tracer over the curve gives then at once the ten coefficients A_n and B_n for $n = 1$ to 5.

Michelson and Stratton analyzer

All the calculating machines and integrators considered so far have been kinematic. We have now to describe a most remarkable instrument based on the equilibrium of a rigid body under the action of springs. The body itself for rigidity's sake is made a hollow cylinder H, shown in fig. 24 in end view. It can turn about its axis, being supported on knife-edges O. To it springs are attached at the prolongation of a horizontal diameter; to the left a series of n small springs s, all alike, side by side at equal intervals at a distance a from the axis of the knife-edges; to the right a single spring S at distance b. These springs are supposed to follow Hooke's law. If the elongation beyond the natural length of a spring is λ, the force asserted by it is $p = k\lambda$. Let for the position of equilibrium l, L be respectively the elongation of a small and the large spring, k, K their constants, then

$$nkla = \mathrm{K}Lb.$$

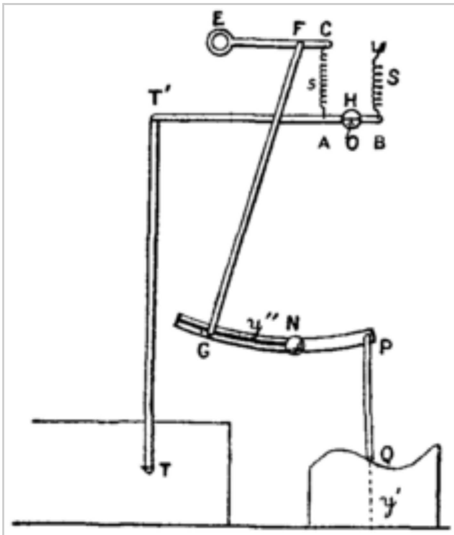

FIG. 24.

The position now obtained will be called the *normal* one. Now let the top ends C of the small springs be raised through distances $y_1, y_2, \ldots y_n$. Then the body H will turn; B will move down through a distance z and A up through a distance $(a/b)z$. The new forces thus introduced will be in equilibrium if

$$ak\left(\sum y - n\frac{a}{b}z\right) = b\mathrm{K}z.$$

Or

$$z = \frac{\sum y}{n\frac{a}{b} + \frac{b}{a}\frac{K}{k}} = \frac{\sum y}{n\left(\frac{a}{b} + \frac{l}{L}\right)}.$$

This shows that the displacement z of B is proportional to the sum of the displacements y of the tops of the small springs. The arrangement can therefore be used for the addition of a number of displacements. The instrument made has eighty small springs, and the authors state that from the experience gained there is no impossibility of increasing their number even to a thousand. The displacement z, which necessarily must be small, can be enlarged by aid of a lever OT′. To regulate the displacements y of the points C (fig. 24) each spring is attached to a lever EC, fulcrum E. To this again a long rod FG is fixed by aid of a joint at F. The lower end of this rod rests on another lever GP, fulcrum N, at a changeable distance $y'' = $ NG from N. The elongation y of any spring s can thus be produced by a motion of P. If P be raised through a distance y', then the displacement y of C will be proportional to $y'y''$; it is, say, equal to $\mu y'y''$ where μ is the same for all springs. Now let the points C, and with it the springs s, the levers, &c., be numbered $C_0, C_1, C_2 \ldots$ There will be a zero-position for the points P all in a straight horizontal line. When in this position the points C will also be in a line, and this we take as axis of x. On it the points $C_0, C_1, C_2 \ldots$ follow at equal distances, say each equal to h. The point C_k lies at the distance kh which gives the x of this point. Suppose now that the rods FG are all set at unit distance NG from N, and that the points P be raised so as to form points in a continuous curve $y' = \varphi(x)$, then the points C will lie in a curve $y = \mu\varphi(x)$. The area of this curve is

$$\mu \int_0^c \varphi(x)dx.$$

Approximately this equals $\sum hy = h\sum y$. Hence we have

$$\int_0^c \varphi(x)dx = \frac{h}{\mu}\sum y = \frac{\lambda h}{\mu}z,$$

where z is the displacement of the point B which can be measured. The curve $y' = \varphi(x)$ may be supposed cut out as a templet. By putting this under the points P the area of the curve is thus determined—the instrument is a simple integrator.

The integral can be made more general by varying the distances NG $= y''$. These can be set to form another curve $y'' = f(x)$. We have now $y = \mu y'y'' = \mu f(x)\,\varphi(x)$, and get as before

$$\int_0^c f(x)\,\varphi(x)dx = \frac{h}{\mu}z.$$

These integrals are obtained by the addition of ordinates, and therefore by an approximate method. But the ordinates are numerous, there being 79 of them, and the results are in consequence very accurate. The displacement z of B is small, but it can be magnified by taking the reading of a point T′ on the lever AB. The actual reading is done at point T connected with T′ by a long vertical rod. At T either a scale can be placed or a drawing-board, on which a pen at T marks the displacement.

If the points G are set so that the distances NG on the different levers are proportional to the terms of a numerical series

$$u_0 + u_1 + u_2 + \ldots$$

and if all P be moved through the same distance, then z will be proportional to the sum of this series up to 80 terms. We get an *Addition Machine*.

The use of the machine can, however, be still further extended. Let a templet with a curve $y' = \varphi(\xi)$ be set under each point P at right angles to the axis of x hence parallel to the plane of the figure. Let these templets form sections of a continuous surface, then each section parallel to the axis of x will form a curve like the old $y' = \varphi(x)$, but with a variable parameter ξ, or $y' = \varphi(\xi, x)$. For each value of ξ the displacement of T will give the integral

$$\mathrm{Y} = \int_0^c f(x)\ \varphi(\xi x)\ dx = \mathrm{F}(\xi), \ldots (1)$$

where Y equals the displacement of T to some scale dependent on the constants of the instrument.

If the whole block of templets be now pushed under the points P and if the drawing-board be moved at the same rate, then the pen T will draw the curve Y = F(ξ). The instrument now is an *integraph* giving the value of a definite integral as function of a *variable parameter*.

Having thus shown how the lever with its springs can be made to serve a variety of purposes, we return to the description of the actual instrument constructed. The machine serves first of all to sum up a series of harmonic motions or to draw the curve

$$\mathrm{Y} = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots (2)$$

The motion of the points P_1P_2 ... is here made harmonic by aid of a series of excentric disks arranged so that for one revolution of the first the other disks complete 2, 3, ... revolutions. They are all driven by one handle. These disks take the place of the templets described before. The distances NG are made equal to the amplitudes a_1, a_2, a_3, ... The drawing-board, moved forward by the turning of the handle, now receives a curve of which (2) is the equation. If all excentrics are turned through a right angle a sine-series can be added up.

It is a remarkable fact that the same machine can be used as a harmonic analyser of a given curve. Let the curve to be analysed be set off along the levers NG so that in the old notation it is

$$y'' = f(x),$$

whilst the curves $y' = \varphi(x\xi)$ are replaced by the excentrics, hence ξ by the angle θ through which the first excentric is turned, so that $y'_k = \cos k\theta$. But $kh = x$ and $nh = \pi$, n being the number of springs s, and π taking the place of c. This makes

$$k\theta = \tfrac{n}{\pi}\theta x.$$

Hence our instrument draws a curve which gives the integral (1) in the form

$$y = \tfrac{2}{\pi} \int_0^\pi f(x) \cos\left(\tfrac{n}{\pi}\theta x\right)\ dx$$

as a function of θ. But this integral becomes the coefficient a_m in the cosine expansion if we make

$$\tfrac{\theta n}{\pi} = m \text{ or } \theta = \tfrac{m\pi}{n}.$$

The ordinates of the curve at the values $\theta = \tfrac{\pi}{n}, \tfrac{2\pi}{n}, \ldots$ give therefore all coefficients up to $m = 80$. The curve shows at a glance which and how many of the coefficients are of importance.

The instrument is described in *Phil. Mag.*, vol. xlv., 1898. A number of curves drawn by it are given, and also examples of the analysis of curves for which the coefficients a_m are known. These indicate that a remarkable accuracy is obtained.

(O. H.)

Endnotes

1. For a fuller description of the manner in which a mere addition machine can be used for multiplication and division, and even for the extraction of square roots, see an article by C.V. Boys in *Nature*, 11th July 1901.